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GEOMETRIC FORMULATION OF EDGE END NODAL FINITE ELEMENT EQUATIONS IN ELECTROMAGNETICS

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Abstract* – Finite element equations for electromagnetic fields are examined, in particular nodal elements using scalar potential formulation and edge elements for vector potential formulation. It is shown how the equations usually obtained via variational approach may be more conveniently derived using integral methods employing a geometrical description of the interpolating functions of edge and facet elements. Moreover, the resultant equations describe the equivalent multi-branch circuit models.

Introduction

Finite element equations are commonly derived using a variational approach, including weak forms (the Galerkin weighted residual method) and/or a strong formulation via a functional (the Rayleigh-Ritz method). These equations have various geometrical interpretations [1] and may be explained using the language of circuit theory by considering the nodal or loop descriptions of equivalent magnetic or electric circuits [2, 3]. The classical equivalent circuits, however, arise from integral formulations. In this work we will demonstrate that by applying appropriate geometrical forms to the interpolating functions of an edge or a facet element the finite element equations may also be derived via integral methods. We will show that by applying approximate integration in the finite element formulation for a mesh with rectangular parallelepiped elements classical expressions for magnetic and electric networks emerge. Both magnetic and electric fields are considered, while for the electric fields conducting and displacement currents may be present.

Geometrical representation of interpolating functions of finite elements

Consider the interpolating functions of an 8-node, 6-facet element (Fig 1a), which allow for the field to be determined at an arbitrary point P within the element. These functions are related to geometrical forms of eight hexahedra v_i defined by drawing straight lines through the point P as shown in Fig. 1. It is well known that the ratio of the volume of the ith hexahedron to the element volume V_e is related to the ith interpolating function of the nodal element, but it is rarely appreciated that the facets and edges of the volume v_i inside the element represent interpolating functions of the associated edge or facet elements. For example, the interpolating function $\boldsymbol{w}_{e4,8}$ of the edge element for the edge P_4P_8 expresses the ratio of the facet vector $\boldsymbol{s}_{4,8}$ to the volume V_e (Fig.1a), while the ratio $\boldsymbol{r}_i/V_e$ describes the interpolating function $\boldsymbol{w}_{fi}$ of the facet element for the facet $\boldsymbol{S}_i$. In a similar manner the interpolating functions for triangular prisms and pentahedron elements may be expressed. As an example, for the pentahedron of Fig. 1b, the expressions for the interpolating functions of the edge element for the edges P_4P_5 and P_2P_5 take the form $\boldsymbol{w}_{e4,5}=\boldsymbol{s}_{4,5}/(2V_e)$ and $\boldsymbol{w}_{e2,5}= \boldsymbol{s}_{2,5}/V_e$, respectively, while the interpolating functions of the facet element for the facets $\boldsymbol{S}_3$ and $\boldsymbol{S}_4$ are given by $w_{f3}=\boldsymbol{r}_3/(2V_e)$ and $w_{f4}=\boldsymbol{r}_4/V_e$, respectively.

By analysing the relevant integrals, where the geometrical forms provided are integrands, it may be easily inferred that the volume integral in V_e of the product of $\boldsymbol{w}_{ei,j}$ and the current density vector, or flux density vector, represents current, or flux, associated with the region next to P_iP_j. At the same time the volume integral of the product of $\boldsymbol{w}_{fi}$ and the magnetic field strength represents the average value of voltage.

In 2D systems, where the z component of the field is absent, e.g. H_z=0, and the other components are functions of x and y only, the element depicted in Fig. 1b is reduced to a triangle with vertices $P_1P_2P_3$. The four-sided facets of the pentahedron then ‘collapse’ to the sides of a triangle, e.g. the facet $\boldsymbol{S}_1$ becomes the edge P_2P_3. The edges P_iP_{i+3}, (i=1, 2, 3) of a prism, parallel to the z axis, are represented in 2D by nodes P_i. The functions of the edge element for these edges P_iP_{i+3} become in 2D the functions of the nodal element for the nodes P_i, while the functions of the facet element for the edges of the prism become similar to the functions of the edge element for the edge of the triangular facets. This similarity arises due to the specific properties of the function describing the pentahedron when the point P lies on the triangular surface of the facet, and thus its z coordinate of point P equals zero. The consequence of this similarity is the fact that the scalar and vector potential formulations in 2D are similar.

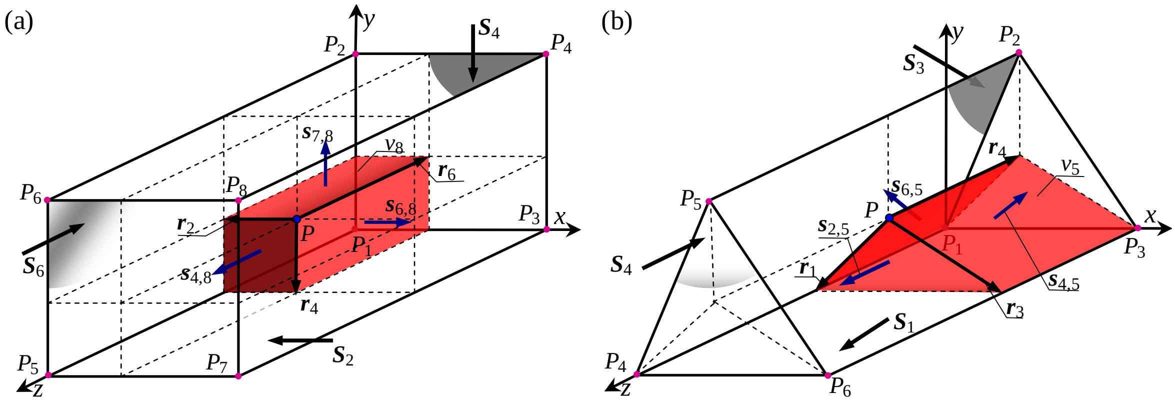

Fig.1. Geometric representation of interpolation functions of hexahedron (a) and 9-edge pentahedron (b).

Integral formulations of the finite element equations

We consider both vector potential ($\boldsymbol{A}$ for the magnetic field and $\boldsymbol{T}$ for the electric field) and scalar potential formulations (Ω for the magnetic field and V for the electric field). It was noted in [2] that the finite element equations formulated for nodal elements and scalar potentials are related to nodal equations of an edge network (EN) with branches assigned to element edges. The equations describing the edge values of vector potentials represent loop (mesh) equations of a facet network (FN) with nodes positioned in element centres and branches passing through the facets. Figure 2 shows the edge and facet models of single hexahedron and pentahedron elements. By appropriate connections between elements we create the network model of the discretised volume. In the case of the edge network we make parallel connections between branches associated with common element edges. The facet network, on the other hand, involves connecting in series the branches of the facet models of elements with a common facet. We first consider the equations for the edge model, that is the scalar potential formulation using nodal elements.

Integral formulations for scalar potentials and nodal elements

A single nodal equation of the edge network results from the summation of currents or fluxes in the branches having common nodes. The currents $i_{i,j}$ and fluxes $\phi_{i,j}$ in the branch P_iP_j associated with the element edge are described by

$$i_{i,j}=\int_{V_e}\boldsymbol{w}_{ei,j}\boldsymbol{J}\,\mathrm{d}v\ ,\qquad \phi_{i,j}=\int_{V_e}\boldsymbol{w}_{ei,j}\boldsymbol{B}\,\mathrm{d}v\ . \tag{1a,b}$$

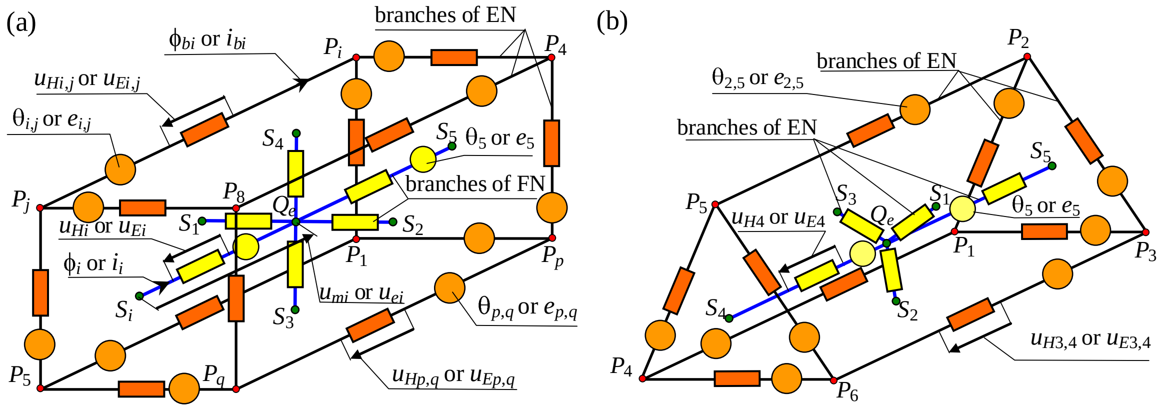

Fig. 2. Edge and facet models of hexahedron (a) and 9-edge pentahedron (b).

Next the constitutive equations are imposed $\boldsymbol{J}$=σ$\boldsymbol{E}$+d(ε$\boldsymbol{E}$)/dt and $\boldsymbol{B}$=μ$\boldsymbol{H}$ and the $\boldsymbol{E}$ and $\boldsymbol{H}$ vectors are expressed in terms of the functions of the edge element, hence

$$i_{i,j}=\sum_{p,q}u_{Ep,q}\int_{V_e}\boldsymbol{w}_{ei,j}\gamma\boldsymbol{w}_{ep,q}\,\mathrm{d}v\ ,\qquad \phi_{i,j}=\sum_{p,q}u_{Hp,q}\int_{V_e}\boldsymbol{w}_{ei,j}\mu\boldsymbol{w}_{ep,q}\,\mathrm{d}v\ . \tag{2a,b}$$

In the above equations the summation includes all edges P_pP_q of the element, where $\gamma=\sigma+p\varepsilon$ (p=d/dt), and $u_{Ep,q}$, $u_{Hp,q}$ are the edge values of vectors $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively. These values represent the voltages on the elements of the equivalent element model (see Fig. 2), $u_{Hp,q}$ is the voltage across the permeance and $u_{Ep,q}$ the voltage across the parallel connection of the capacitance and conductance. The relationship (2) suggests that in the equivalent models of the element there exist mutual couplings between branches, as in the expressions for current and flux in the branch P_iP_j we have not only voltages in this branch but also in other branches of the element. Consequently the mass matrices for nodal element method are non-diagonal.

By using the substitutions

$$\mathrm{grad}\,V=\boldsymbol{E}+\mathrm{d}\boldsymbol{A}/\mathrm{d}t\boldsymbol{T}\ ,\qquad \mathrm{grad}\,\Omega=\boldsymbol{H}-\boldsymbol{T}\ , \tag{3a,b}$$

the edge values of vectors $\boldsymbol{H}$ and $\boldsymbol{E}$ may be expressed as

$$u_{Ep,q}=V_q-V_p+e_{p,q}\ ,\qquad u_{Hp,q}=\Omega_q-\Omega_p+\theta_{p,q}\ , \tag{4a,b}$$

where Ω_q, Ω_p, V_q, V_p are the nodal values of scalar potential for nodes P_q, P_p, and $\theta_{p,q}$ and $e_{p,q}$ are the edge values of $\boldsymbol{T}$ and –d$\boldsymbol{A}$/dt, respectively, and represent the sources of magnetic and electric field. Substituting (4) to (2) yields

$$i_{i,j}=i_{si,j}+\sum_{p,q}(V_q-V_p)\int_{V_e}\boldsymbol{w}_{ei,j}\gamma\boldsymbol{w}_{ep,q}\,\mathrm{d}v\ ,\quad \phi_{i,j}=\phi_{si,j}+\sum_{p,q}(\Omega_q-\Omega_p)\int_{V_e}\boldsymbol{w}_{ei,j}\mu\boldsymbol{w}_{ep,q}\,\mathrm{d}v\ , \tag{5a,b}$$

where $i_{si,j}$ is the current source and $\phi_{si,j}$ is the flux source related to branch P_iP_j,

$$i_{si,j}=\sum_{p,q}e_{p,q}\int_{V_e}\boldsymbol{w}_{ei,j}\gamma\boldsymbol{w}_{ep,q}\,\mathrm{d}v\ ,\quad \phi_{si,j}=\sum_{p,q}\theta_{p,q}\int_{V_e}\boldsymbol{w}_{ei,j}\mu\boldsymbol{w}_{ep,q}\,\mathrm{d}v\ . \tag{6a,b}$$

It has already been noted that a single equation of the nodal element formulation may be related to a nodal equation of the edge network and is found by equalling to zero the sum of currents or fluxes in branches with a common node. For the node P_i we may therefore write

$$\sum_j i_{i,j} = 0\,, \qquad \sum_j \phi_{i,j} = 0\,. \tag{7a,b}$$

where j is the node index P_j (j=1,2,..n) of all n branches P_iP_j containing the node P_i.
Substituting (5) into (7) and some further manipulation results in finite element formulation in terms of the scalar potential, which may be written as

$$\sum_j \sum_{p,q} (V_q - V_p) \int_{V_{ej}} \boldsymbol{w}_{ei,j} \gamma \boldsymbol{w}_{ep,q} \, \mathrm{d}v = -\sum_j i_{si,j}\,, \quad \sum_j \sum_{p,q} (\Omega_q - \Omega_p) \int_{V_{ej}} \boldsymbol{w}_{ei,j} \mu \boldsymbol{w}_{ep,q} \, \mathrm{d}v = -\sum_j \theta_{si,j}\,, \tag{8a,b}$$

where V_{ej} is the volume of the element containing the edge P_iP_j.

Integral formulations for vector potentials and edge elements

The derivations of the last section referred to the final element equations formulated in terms of the scalar potentials. The substitutions (3) are also used in the vector potential formulations. A volume integral then needs to be considered of the products of the edge element functions and relevenat terms in (3). Both sides of (3) are multiplied by a function $\boldsymbol{w}_{fi}$ of the facet element for the ith facet and the resultant expressions are integrated over the element volume, which ultimately leads to

$$\int_{V_e} \boldsymbol{w}_{fi} \mathrm{grad}\Omega \mathrm{d}v = \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{H} \mathrm{d}v - \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{T} \mathrm{d}v\,, \quad \int_{V_e} \boldsymbol{w}_{fi} \mathrm{grad} V \mathrm{d}v = \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{E} \mathrm{d}v + \int_{V_e} \boldsymbol{w}_{fi} (\mathrm{d}\boldsymbol{A}/\mathrm{d}t) \mathrm{d}v\,. \tag{9a,b}$$

It can be shown that

$$\int_{V_e} \boldsymbol{w}_{fi} \,\mathrm{grad}\,\Omega \mathrm{d}v = \Omega_{Si} - \Omega_Q, \qquad \int_{V_e} \boldsymbol{w}_{fi} \,\mathrm{grad}\, V \,\mathrm{d}v = V_{Si} - V_Q\,, \tag{10a,b}$$

where Ω_{Si} and V_{Si} are average values of potentials Ω and V for the ith facet, respectively, assigned in Fig. 2 to the node S_i, while Ω_Q and V_Q denote average values of potentials Ω and V in the element, associated with the node Q_e. Through using (10) the relationship (9) may be written as

$$\Omega_{Si} - \Omega_Q = u_{mi} = u_{Hi} - \theta_i\,, \qquad V_{Si} - V_Q = u_{ei} = u_{Ei} - e_i\,, \tag{11a,b}$$

where

$$u_{Hi} = \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{H} \mathrm{d}v\,, \qquad u_{Ei} = \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{E} \mathrm{d}v\,, \tag{12a,b}$$

$$\theta_i = \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{T} \mathrm{d}v\,, \qquad e_i = -\mathrm{d}\left(\int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{A} \mathrm{d}v\right) \big/ \mathrm{d}t\,. \tag{13a,b}$$

The equations (11) describe inter-nodal voltages u_{mi}, u_{ei} for the branch Q_eS_i of the facet network (see Fig 2). The terms u_{Hi} and u_{Ei} represent voltages across the reluctance and across the impedance of the given branch, respectively, whereas θ_I and e_i are the branch magnetomotive force (*mmf*) and the electromotive force (*emf*).

Analysing the geometrical form describing the function of the facet element reveals that the integrals in (13), describing the *mmf* and *emf*, may be viewed as the edge values of the potentials $\boldsymbol{T}$ and $\boldsymbol{A}$ for the edge described by a vector $\boldsymbol{r}_i$ which starts at the centre S_i of the facet and ends in the middle Q_e of the element. The mentioned edge values may be treated as loop currents and fluxes in a mesh of the edge network, associated with the facet $\boldsymbol{S}_i$. For example, in the model of Fig. 2a such a loop is made up of the branches with nodes $P_jP_qP_8P_5$.

When formulating the expressions for voltages u_{Hi} and u_{Ei}, across the reluctance and impedance, the constitutive equations are used $\boldsymbol{H}=\mu^{-1}\boldsymbol{B}$ and $\boldsymbol{E}=\gamma^{-1}\boldsymbol{J}$, whereas to describe the flux density $\boldsymbol{B}$ and current density $\boldsymbol{J}$ the functions of the facet element are used. Substituting yields

$$u_{Hi}=\sum_q \phi_q \int_{V_e} \boldsymbol{w}_{fi}\mu^{-1}\boldsymbol{w}_{fq}\mathrm{d}v\,, \qquad u_{Ei}=\sum_q i_q \int_{V_e} \boldsymbol{w}_{fi}\gamma^{-1}\boldsymbol{w}_{fq}\mathrm{d}v \tag{14a,b}$$

The summation above refers to all the facets of the element, the integrals (under the summation) describe equivalent reluctances and impedances, respectively, while ϕ_q and i_q are the facet values of the flux density $\boldsymbol{B}$ and current density $\boldsymbol{J}$ for the qth facet, respectively. Expression (14) suggests that in the facet model of the element couplings between branches may exist, as indeed was the case for the edge model. Consequently the mass matrices for edge element equations are also not diagonal, as is the case for the nodal elements.

When setting up the equations for the edge element method the facet quantities are expressed in terms of the edge values. The following relationships hold for the qth facet

$$\phi_q=\sum_{r,j}\varphi_{r,j}\,, \qquad i_q=\sum_{r,j} i_{or,j}\,, \tag{15a,b}$$

where $\varphi_{r,j}$ and $i_{or,j}$ are the edge values of $\boldsymbol{A}$ and $\boldsymbol{T}$, respectively, for the edge P_rP_j of the facet $\boldsymbol{S}_q$. For example, after applying (15a) to the facet $\boldsymbol{S}_1$ of Fig. 1b we find $\phi_1=\varphi_{2,3}+\varphi_{3,6}+\varphi_{6,5}+\varphi_{5,2}$. The edge values of the potentials $\boldsymbol{A}$ and $\boldsymbol{T}$ represent the fluxes and currents in loops around the edge, that is in the loops of the facet network. Therefore (15) expresses the flux/current in the qth branch of the facet network in terms of fluxes/currents in loops around element edges.

The equation of the edge element method for the edge P_iP_j is found by summing up the voltages u_{mi} and u_{ei} for all branches Q_eS_i around that edge, that is for those branches for which the node S_i is related to the facet $\boldsymbol{S}_i$ with the edge P_iP_j. The resultant sums are then equated to zero

$$\sum_i u_{mi}=0\,, \qquad \sum_i u_{ei}=0 \tag{16a,b}$$

Incorporating (11) to (15) into the above results in

$$\sum_i\sum_q\sum_{r,j}\varphi_{r,j}\int_{V_{ei}}\boldsymbol{w}_{fi}\mu^{-1}\boldsymbol{w}_{fq}\mathrm{d}v=\sum_i\theta_i\,, \qquad \sum_i\sum_q\sum_{r,j} i_{or,j}\int_{V_{ei}}\boldsymbol{w}_{fi}\gamma^{-1}\boldsymbol{w}_{fq}\mathrm{d}v=\sum_i e_i\,. \tag{17a,b}$$

The right hand sides of these equations represent the resultant loop *mmf* $\theta_{or,j}$ and loop *emf* $e_{or,j}$. These resultant *mmf* and *emf* may be established (a) from edge values of $\boldsymbol{T}$ and d$\boldsymbol{A}$/dt, or (b) facet values of $\boldsymbol{J}$ and d$\boldsymbol{B}$/dt. In the former case the potentials $\boldsymbol{T}$ and $\boldsymbol{A}$ in (13) are expressed in terms of their edge values. After substitution we find

$$\theta_i=\sum_{r,j} i_{or,j}\int_{V_e}\boldsymbol{w}_{fi}\boldsymbol{w}_{er,j}\,\mathrm{d}v\,, \qquad e_i=-\mathrm{d}\Big(\sum_{r,j}\varphi_{r,j}\int_{V_e}\boldsymbol{w}_{fi}\boldsymbol{w}_{er,j}\,\mathrm{d}v\Big)\Big/\mathrm{d}t\,. \tag{18a,b}$$

The integral in (18) is dimensionless and may be treated as the weight parameter $\vartheta_{r,j}^{(i)}$

$$\vartheta_{r,j}^{(i)}=\int_{V_e}\boldsymbol{w}_{fi}\boldsymbol{w}_{er,j}\,\mathrm{d}v\,, \tag{19}$$

This parameter defines the weight of the loop current/flux around the edge P_rP_j (in the loop of the facet network) in the expression describing the current/flux in the loop of the edge network associated with the facet $\boldsymbol{S}_i$. The parameter $\vartheta^{(i)}_{r,j}$ is also used in the above mentioned approach (b) where the loop *mmf*s and *emf*s are established on the basis of the facet values, leading to

$$\theta_{or,j} = \sum_q \vartheta^{(q)}_{r,j} i_q \ , \quad e_{or,j} = -\mathrm{d}(\sum_q \vartheta^{(q)}_{r,j} \phi_q)/\mathrm{d}t \tag{20a,b}$$

The summation here refers to all facets $\boldsymbol{S}_q$ of elements sharing the common edge P_rP_j, while the parameter $\vartheta^{(q)}_{r,j}$ is a weight with which the current/flux passing through the loop of the edge network – hence through the facet $\boldsymbol{S}_q$ – is taken when the current/flux passing through the loop of the facet network (the loop around the edge P_rP_j) is calculated. As may be seen in Fig. 1 the significant proportion of the vectors representing the functions of the facet element is perpendicular to the vectors representing the functions of th edge elements. Thus the scalar product $\boldsymbol{w}_{fi}\boldsymbol{w}_{er,j}$ of such vectors equals zero and the resulting weight parameter is also zero. For the element of Fig. 1a the only non-zero (and in fact equal to 1/8) weights are those which related to transformations between quantities related to facets and edges with parallel vectors.

All the component terms discussed above appearing in the equations for the nodal element method using scalar potential formulations and the edge element method using vector potentials formulations are tabulated and presented in Table 1.

Table 1. Components of the FEM equations

Potential, type of network	FE equations	FE equation components	Description of the integrands	FE coefficients	Substitutions, entries to RHS vector
Ω, Edge-magnetic	$\sum\phi_{i,j}=0$	$\phi_{i,j}=\int_{V_e}\boldsymbol{w}_{ei,j}\boldsymbol{B}\mathrm{d}v$	$\boldsymbol{B}=\mu\boldsymbol{H}=\mu\sum\boldsymbol{w}_{ep,q}u_{Hp,q}$	$\int_{V_e}\boldsymbol{w}_{ei,j}\mu\boldsymbol{w}_{ep,q}\mathrm{d}v$	$u_{Hp,q}=\Omega_q-\Omega_p+\theta_{p,q}$, $\sum\theta_{p,q}$
V, Edge-electric	$\sum i_{i,j}=0$	$i_{i,j}=\int_{V_e}\boldsymbol{w}_{ei,j}\boldsymbol{J}\,\mathrm{d}v$	$\boldsymbol{J}=\nu\boldsymbol{E}=\gamma\sum\boldsymbol{w}_{ep,q}u_{Ep,q}$	$\int_{V_e}\boldsymbol{w}_{ei,j}\gamma\boldsymbol{w}_{ep,q}\mathrm{d}v$	$u_{Ep,q}=V_q-V_p+e_{p,q}$, $\sum e_{p,q}$
$\boldsymbol{A}$, Facet-magnetic	$\sum u_{mi}=0$	$u_{mi}=\int_{V_e}\boldsymbol{w}_{fi}\mathrm{grad}\Omega\mathrm{d}v$	$\mathrm{grad}\Omega=\nu\boldsymbol{B}-\boldsymbol{T}=\nu\sum w_{fq}\phi_q-\boldsymbol{T}$	$\int_{V_e}\boldsymbol{w}_{fi}\nu\boldsymbol{w}_{fq}\mathrm{d}v$	$\phi_q=\sum\varphi_{r,j}$, $\sum\int_{V_e}w_{fi}\boldsymbol{T}$
$\boldsymbol{T}$, Facet-electric	$\sum u_{ei}=0$	$u_{ei}=\int_{V_e}\boldsymbol{w}_{fi}\mathrm{grad}V\mathrm{d}v$	$\mathrm{grad}V=\rho\boldsymbol{E}+\mathrm{p}\boldsymbol{A}=\rho\sum w_{fq}i_q+\mathrm{p}\boldsymbol{A}$	$\int_{V_e}\boldsymbol{w}_{fi}\rho\boldsymbol{w}_{fq}\mathrm{d}v$	$i_q=\sum i_{or,j}$, $\sum\int_{V_e}w_{fi}\mathrm{p}\boldsymbol{A}$

Comments: $\phi_{i,j}$, $i_{i,j}$ are fluxes and currents associated with edges P_iP_j of common node; $u_{Hp,q}$, $u_{Ep,q}$ are edge values $\boldsymbol{H}$ and $\boldsymbol{E}$ for edges P_pP_q; u_{mi}, u_{ei} are voltages associated with facets S_i of common edge; ϕ_q, i_q are facet values of $\boldsymbol{B}$ and $\boldsymbol{J}$ expressed by edge values of $\boldsymbol{A}$ and $\boldsymbol{T}$; $\varphi_{i,j}$, $i_{oi,j}$ are edge values of $\boldsymbol{A}$ and $\boldsymbol{T}$ for P_iP_j; $\gamma=\sigma+\mathrm{p}\varepsilon$; $\mathrm{p}=\mathrm{d}/\mathrm{d}t$;$\rho=1/\gamma$

Approximate description of integrals in FE equations

The integrals describing coefficients of the finite element method may be easily established using accurate analytical methods, but only for regular multi-sided elements and linear materials. Irregular elements and non-linear material properties inevitably call for approximate numerical methods. The authors of the article recommend the following approximation

$$\int_{V_e} f\,\mathrm{d}v = \frac{V_e}{n_n}\sum_{i=1}^{n_n} f(P_i)\,, \tag{21}$$

where n_n is the number of element nodes P_i, and $f(P_i)$ is the value of the function f in the node P_i [5]. Application of this approximation results in significant simplification of the description of the coefficients of a mesh made up of parallelepiped elements. As an example, consider the coefficients of the equations describing the magnetic field in a magnetically non-linear region. When a scalar potential formulation Ω is employed the equation coefficients represent the permeance

$$\Lambda_{i,j}^{(p,q)} = \int_{V_e} \boldsymbol{w}_{ei,j} \mu \boldsymbol{w}_{ep,q} \mathrm{d}v \ , \tag{22}$$

whereas if a vector potential $\boldsymbol{A}$ is used they represent the reluctance

$$R_i^{(q)} = \int_{V_e} \boldsymbol{w}_{fi} \nu \boldsymbol{w}_{fq} \mathrm{d}v \ , \tag{23}$$

where $\mu=\mu(\boldsymbol{H})$ and $\nu=\mu^{-1}=\nu(\boldsymbol{B})$.

Applying (21) yields

$$\Lambda_{i,j}^{(i,j)} = \mu_{i,j} V_e / l_{i,j}^2 \ , \quad \Lambda_{i,j}^{(p,q)} = 0 \ \text{for } i,j \neq p,q \ , \tag{24a,b}$$

$$R_i^{(i)} = \nu_i V_e / S_i^2 \ , \quad R_i^{(q)} = 0 \ \text{for } i \neq q, \tag{25a,b}$$

where $l_{i,j}$ is the length of the edge P_iP_j, S_i is the surface area of the facet $\boldsymbol{S}_i$, $\mu_{i,j}$ describes the magnetic permeability in the proximity of the edge P_iP_j, and ν_i is the reluctivity of the medium in the area close to the facet S_i. For a system with magnetic non-linear characteristics

$$\mu_{i,j} = 0.5\big(\mu(\boldsymbol{H}_i) + \mu(\boldsymbol{H}_j)\big) , \quad \nu_i = 0.25 \sum_{p=1}^{4} \nu(B_p) \ , \tag{26a,b}$$

where $\boldsymbol{H}_i$ is the field intensity in the proximity of the node P_i of the edge P_iP_j, and $\boldsymbol{B}_p$ the flux density in the pth corner P_p of the facet $\boldsymbol{S}_i$. $\boldsymbol{H}_i$ may be found from the edge values of $\boldsymbol{H}$, while $\boldsymbol{B}_p$ from the facet values of $\boldsymbol{B}$. For example, for Figs 1a and 3 the value of $\boldsymbol{H}_i$ in the surrounding region close to P_i and the flux density $\boldsymbol{B}_p$ in the proximity of P_q may be calculated as

$$\boldsymbol{H}_{i,} = \mathbf{1}_x u_{Hi,p} / h_x + \mathbf{1}_y u_{H1,i} / h_y + \mathbf{1}_z u_{Hi,j} / h_z \tag{27}$$

$$\boldsymbol{B}_p = \mathbf{1}_x \phi_2 / (-h_y h_z) + \mathbf{1}_y \phi_4 / (-h_x h_z) + \mathbf{1}_z \phi_i / (h_x h_y) \tag{28}$$

When deriving the above equations it has been taken into account that the facet vectors are directed 'into' the element, whereas the direction of the edges depends on the sequence of the indices in the description of the edge value (see Figs 1a and 2a).

A closer inspection of (24) and (25) reveals that in the models following from application of the recommended approximating formula (21) there are no couplings between branches; hence the resulting mass matrix is diagonal and the coefficients express only self permeances/reluctances. Expressions (24a) and (25a) describing these permeances/reluctances are identical to those obtained form a classical formulation using magnetic networks, whose parameters may be established via the tubes and slices method [7]. For a hexahedron the geometrical forms are as shown in Fig. 3; it is taken into account that the permeance of the branch P_iP_j of the edge network is related to the 'magnetic conductance' of a block of length $l_{i,j}$ and cross section $0.25h_xh_y$ (Fig. 3a). The reluctance in the branch Q_eS_i of the facet network, on the other hand, represents the 'magnetic resistance' of a block of length $0.5l_{i,j}$ and cross section h_xh_y.

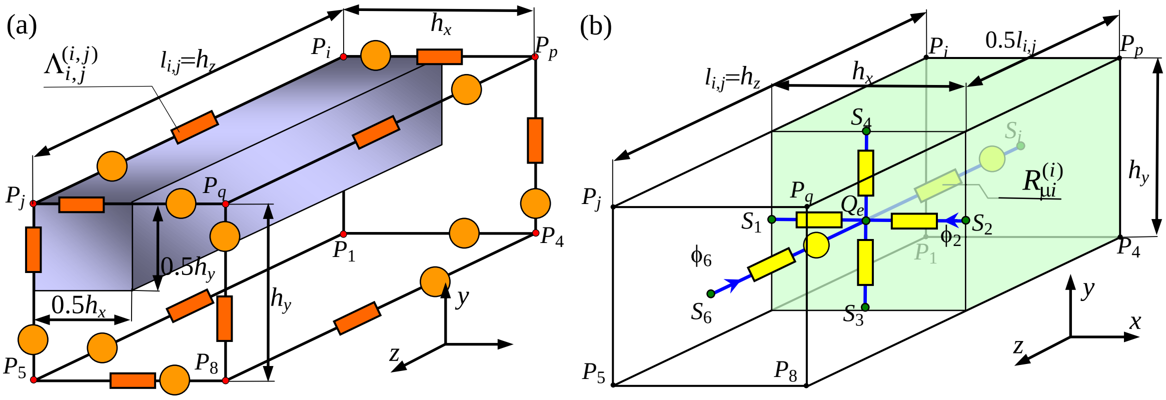

Fig. 3. Geometric forms representing permeance (a) and reluctance (b).

Conclusion

Finite element equations may be derived from equivalent circuit models without the need for variational formulation. By exploiting the geometrical properties of interpolating functions the relevant parameters may be established using integral methods. The presented approach is valid not only for hexahedra and pentahedra but also for tetrahedra [1] and mixed finite elements [6]. The proposed approach promises to be very beneficial in teaching, especially to students well familiar with circuit methods, to whom the analogy of the finite element formulation to loop or nodal magnetic or electric networks may be appealing and easier to understand. Thus the teaching of computational electromagnetics may be seen as supplementing the circuit theory by the relevant information about the integral methods of calculating network model parameters, as argued in this paper. The presented methods are also helpful when formulating classical network models, such as described in [4].

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